

Factorization method for difference equations of hypergeometric type on nonuniform lattices *

R. Álvarez-Nodarse^{†‡} and R. S. Costas-Santos[†]

[†] Departamento de Análisis Matemático.

Universidad de Sevilla. Apdo. 1160, E-41080 Sevilla

[‡] Instituto Carlos I de Física Teórica y Computacional,

Universidad de Granada, E-18071 Granada, Spain

24th May 2001

Abstract

We study the factorization of the hypergeometric-type difference equation of Nikiforov and Uvarov on nonuniform lattices. An explicit form of the raising and lowering operators is derived and some relevant examples are given.

1 Introduction

In this paper we will deal with the so-called factorization method (FM) of the hypergeometric-type difference equations on nonuniform lattices. The FM was already used by Darboux [14] and Schrödinger [25, 26] to obtain the solutions of differential equations, and also by Infeld and Hull [16] for finding analytical solutions of certain classes of second order differential equations. Later on, Miller extended it to difference equations [20] and q -differences –in the Hahn sense– [21]. For more recent works see e.g. [4, 10, 11, 29, 30] and references therein.

The classical FM was based on the existence of a so-called raising and lowering operators for the corresponding equation that allows to find the explicit solutions in a very easy way. Going further, Atakishiyev and coauthors [4, 6, 10] have found the dynamical symmetry algebra related with the FM and the differential or difference equations. Of special interest was the paper by Smirnov [27] in which the equivalence of the FM and the Nikiforov et al theory [23] was shown, furthermore this paper pointed out that the aforementioned equivalence remains valid also for the nonuniform lattices that was shown later on in [28, 29]. In particular, in [29] a detailed study of the FM and its equivalence with the Nikiforov et al. approach to difference equations [23] have been established. Also, in [12], a special nonuniform lattice was considered. In fact, in [12] the author constructed the FM for the Askey–Wilson polynomials using basically the difference equation for the polynomials. In the present paper we will continue the research of the nonuniform lattice case. In fact, following the idea by Bangerezako [12] for the Askey–Wilson polynomials and Lorente [18] for the classical continuous and discrete cases, we will obtain the FM for the general polynomial solutions of the hypergeometric difference equation on the general quadratic nonuniform lattice $x(s) = c_1 q^s + c_2 q^{-s} + c_3$.

*Submitted to J. Phys. A: Math. Gen.

We will use, as it is already suggested in [6, 27], not the polynomial solutions but the corresponding normalized functions which is more natural and useful. In such a way, the method proposed here is the generalization of [12] and [18] to the aforementioned nonuniform lattice.

The structure of the paper is as follows. In Section 2 we present some well-known results on orthogonal polynomials on nonuniform lattices [7, 23, 24], in section 3 we introduce the normalized functions and obtain some of their properties such as the lowering and raising operator that allow us, in Section 4, to obtain the factorization for the second order difference equation satisfied by such functions. Finally, in Section 5, some relevant examples are worked out.

2 Some basic properties of the q -polynomials

Here, we will summarize some of the properties of the q -polynomials useful for the rest of the work. For further information see e.g. [23].

We will deal here with the *second order difference equation of the hypergeometric type*

$$\sigma(s) \frac{\Delta}{\Delta x(s - \frac{1}{2})} \left[\frac{\nabla y(s)}{\nabla x(s)} \right] + \tau(s) \frac{\Delta y(s)}{\Delta x(s)} + \lambda y(s) = 0, \quad (1)$$

$$\sigma(s) = \tilde{\sigma}(x(s)) - \frac{1}{2} \tilde{\tau}(x(s)) \Delta x(s - \frac{1}{2}), \quad \tau(s) = \tilde{\tau}(x(s)),$$

where $\nabla f(s) = f(s) - f(s - 1)$ and $\Delta f(s) = f(s + 1) - f(s)$ denote the backward and forward finite difference derivatives, respectively, $\tilde{\sigma}(x(s))$ and $\tilde{\tau}(x(s))$ are polynomials in $x(s)$ of degree at most 2 and 1, respectively, and λ is a constant. In the following, we will use the following notation for the coefficients in the power expansions in $x(s)$ of $\tilde{\sigma}(s)$ and $\tilde{\tau}(s)$

$$\tilde{\sigma}(s) \equiv \tilde{\sigma}[x(s)] = \frac{\tilde{\sigma}''}{2} x^2(s) + \tilde{\sigma}'(0)x(s) + \tilde{\sigma}(0), \quad \tilde{\tau}(s) \equiv \tilde{\tau}[x(s)] = \tilde{\tau}'x(s) + \tilde{\tau}(0). \quad (2)$$

An important property of the above equation is that the k -order *difference derivative* of a solution $y(s)$ of (1), defined by

$$y_k(s)_q = \frac{\Delta}{\Delta x_{k-1}(s)} \frac{\Delta}{\Delta x_{k-2}(s)} \cdots \frac{\Delta}{\Delta x(s)} y(s) \equiv \Delta^{(k)} y(s),$$

also satisfies a difference equation of the hypergeometric type

$$\sigma(s) \frac{\Delta}{\Delta x_k(s - \frac{1}{2})} \left[\frac{\nabla y_k(s)_q}{\nabla x_k(s)} \right] + \tau_k(s) \frac{\Delta y_k(s)_q}{\Delta x_k(s)} + \mu_k y_k(s)_q = 0, \quad (3)$$

where $x_k(s) = x(s + \frac{k}{2})$ and [23, page 62, Eq. (3.1.29)]

$$\tau_k(s) = \frac{\sigma(s+k) - \sigma(s) + \tau(s+k) \Delta x(s+k - \frac{1}{2})}{\Delta x_{k-1}(s)}, \quad \mu_k = \lambda + \sum_{m=0}^{k-1} \frac{\Delta \tau_m(s)}{\Delta x_m(s)}. \quad (4)$$

It is important to notice that the above difference equations have polynomial solutions of the hypergeometric type iff $x(s)$ is a function of the form [7, 24]

$$x(s) = c_1(q)q^s + c_2(q)q^{-s} + c_3(q) = c_1(q)[q^s + q^{-s-\mu}] + c_3(q), \quad (5)$$

where c_1, c_2, c_3 and $q^\mu = \frac{c_1}{c_2}$ are constants which, in general, depend on q [23, 24]. For the above lattice, a straightforward calculation shows that $\tau_k(s)$ is a polynomial of first degree in $x_k(s)$ of the form (see e.g. [7])

$$\begin{aligned}\tau_k(s) &= \tilde{\tau}'_k x_k(s) + \tilde{\tau}_k(0), & \tilde{\tau}'_k &= [2k]_q \frac{\tilde{\sigma}''}{2} + \alpha_q(2k) \tilde{\tau}', \\ \tilde{\tau}_k(0) &= \frac{c_3 \tilde{\sigma}''}{2} (2[k]_q - [2k]_q) + \tilde{\sigma}'(0) [k]_q + c_3 \tau'(\alpha_q(k) - \alpha_q(2k)) + \tilde{\tau}(0) \alpha_q(k),\end{aligned}\quad (6)$$

where the q -numbers $[k]_q$ and $\alpha_q(k)$ are defined by

$$[k]_q = \frac{q^{\frac{k}{2}} - q^{-\frac{k}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \quad \alpha_q(k) = \frac{q^{\frac{k}{2}} + q^{-\frac{k}{2}}}{2}, \quad (7)$$

and $[n]_q!$ are the q -factorials $[n]_q! = [1]_q [2]_q \cdots [n]_q$.

Both difference equations (1) and (3) can be rewritten in the symmetric form

$$\frac{\Delta}{\Delta x(s - \frac{1}{2})} \left[\sigma(s) \rho(s) \frac{\nabla y(s)}{\nabla x(s)} \right] + \lambda_n \rho(s) y(s) = 0,$$

and

$$\frac{\Delta}{\Delta x_k(s - \frac{1}{2})} \left[\sigma(s) \rho_k(s) \frac{\nabla y_k(s)}{\nabla x_k(s)} \right] + \mu_k \rho_k(s) y_k(s) = 0,$$

where $\rho(s)$ and $\rho_k(s)$ are the weight functions satisfying the Pearson-type difference equations

$$\frac{\Delta}{\Delta x(s - \frac{1}{2})} [\sigma(s) \rho(s)] = \tau(s) \rho(s), \quad \frac{\Delta}{\Delta x_k(s - \frac{1}{2})} [\sigma(s) \rho_k(s)] = \tau_k(s) \rho_k(s), \quad (8)$$

respectively. In [23] it is shown that the polynomial solutions of (3) (and so the polynomial solutions of (1)) are determined by the q -analogue of the Rodrigues formula on the nonuniform lattices

$$\frac{\Delta}{\Delta x_{k-1}(s)} \cdots \frac{\Delta}{\Delta x(s)} P_n(x(s))_q \equiv \Delta^{(k)} P_n(x(s))_q = \frac{A_{n,k} B_n}{\rho_k(s)} \nabla_k^{(n)} \rho_n(s), \quad (9)$$

where

$$\begin{aligned}\nabla_k^{(n)} f(s) &= \frac{\nabla}{\nabla x_{k+1}(s)} \frac{\nabla}{\nabla x_{k+2}(s)} \cdots \frac{\nabla}{\nabla x_n(s)} f(s), \\ A_{n,k} &= \frac{[n]_q!}{[n-k]_q!} \prod_{m=0}^{k-1} \left\{ \alpha_q(n+m-1) \tilde{\tau}' + [n+m-1]_q \frac{\tilde{\sigma}''}{2} \right\}\end{aligned}\quad (10)$$

Thus [23, page 66, Eq. (3.2.19)]

$$P_n(x(s))_q = \frac{B_n}{\rho(s)} \nabla^{(n)} \rho_n(s), \quad \nabla^{(n)} \equiv \frac{\nabla}{\nabla x_1(s)} \frac{\nabla}{\nabla x_2(s)} \cdots \frac{\nabla}{\nabla x_n(s)}, \quad (11)$$

where $\rho_n(s) = \rho(s+n) \prod_{k=1}^n \sigma(s+k)$ and

$$\lambda_n = -[n]_q \left\{ \alpha_q(n-1) \tilde{\tau}' + [n-1]_q \frac{\tilde{\sigma}''}{2} \right\}. \quad (12)$$

In this paper we will deal with orthogonal q -polynomials and functions. It can be proven [23], by using the difference equation of hypergeometric-type (1), that if the boundary condition

$$\sigma(s)\rho(s)x^k\left(s - \frac{1}{2}\right)\Big|_{s=a,b} = 0, \quad \forall k \geq 0, \quad (13)$$

holds, then the polynomials $P_n(s)_q$ are orthogonal, i.e.,

$$\sum_{s=a}^{b-1} P_n(x(s))_q P_m(x(s))_q \rho(s) \Delta x\left(s - \frac{1}{2}\right) = \delta_{nm} d_n^2, \quad s = a, a+1, \dots, b-1, \quad (14)$$

where $\rho(s)$ is a solution of the Pearson-type equation (8). In the special case of the linear exponential lattice $x(s) = q^s$ the above relation can be written in terms of the Jackson q -integral (see e.g. [15, 17]) $\int_{z_1}^{z_2} f(t) d_q t$, defined by

$$\int_{z_1}^{z_2} f(t) d_q t = \int_0^{z_2} f(t) d_q t - \int_0^{z_1} f(t) d_q t,$$

where

$$\int_0^z f(t) d_q t = z(1-q) \sum_{k=0}^{\infty} f(zq^k) q^k, \quad 0 < q < 1,$$

as follows:

$$\int_{q^a}^{q^b} P_n(t)_q P_m(t)_q \omega(t) d_q t = \delta_{nm} q^{1/2} d_n^2, \quad t = q^s, \quad \omega(t) \equiv \omega(q^t) = \rho(t). \quad (15)$$

Notice that the above boundary condition (13) is valid for $k = 0$. Moreover, if we assume that a is finite, then (13) is fulfilled at $s = a$ providing that $\sigma(a) = 0$ [23, §3.3, page 70]. In the following we will assume that this condition holds. The squared norm in (14) is given by [23, Chapter 3, Section 3.7.2, pag. 104]

$$d_n^2 = (-1)^n A_{n,n} B_n^2 \sum_{s=a}^{b-n-1} \rho_n(s) \Delta x_n\left(s - \frac{1}{2}\right).$$

There is also a so-called continuous orthogonality. In fact, if there exist a contour Γ such that

$$\int_{\Gamma} \Delta[\rho(z)\sigma(z)x^k\left(z - \frac{1}{2}\right)] dz = 0, \quad \forall k \geq 0, \quad (16)$$

then [23]

$$\int_{\Gamma} P_n(x(z))_q P_m(x(z))_q \rho(z) \Delta x\left(z - \frac{1}{2}\right) dz = 0, \quad n \neq m.$$

A simple consequence of the orthogonality is the following three term recurrence relation:

$$x(s)P_n(x(s))_q = \alpha_n P_{n+1}(x(s))_q + \beta_n P_n(x(s))_q + \gamma_n P_{n-1}(x(s))_q, \quad (17)$$

where α_n , β_n and γ_n are constants. If $P_n(s)_q = a_n x^n(s) + b_n x^{n-1}(s) + \dots$, then using (17) we find

$$\alpha_n = \frac{a_n}{a_{n+1}}, \quad \beta_n = \frac{b_n}{a_n} - \frac{b_{n+1}}{a_{n+1}}, \quad \gamma_n = \frac{a_{n-1}}{a_n} \frac{d_n^2}{d_{n-1}^2}. \quad (18)$$

To obtain the explicit values of α_n , β_n we will use the following lemma, –interesting in its own right– that can be proven by induction:

Lemma 2.1

$$\Delta^{(k)}x^n(s) = \frac{[n]_q!}{[n-k]_q!}x_k^{n-k}(s) + c_3 \left(n \frac{[n-1]_q!}{[n-k-1]_q!} - (n-k) \frac{[n]_q!}{[n-k]_q!} \right) x_k^{n-k-1}(s) + \dots.$$

In the case $k = n - 1$, it becomes

$$\Delta^{(n-1)}x^n(s) = [n]_q!x_{n-1}(s) + c_3[n-1]_q!(n - [n]_q). \quad (19)$$

Now, using the Rodrigues formula (9) for $k = n - 1$,

$$\Delta^{(n-1)}P_n(x(s))_q = \frac{A_{n,n-1}B_n}{\rho_{n-1}(s)} \nabla_{n-1}^{(n)}\rho_n(s) = \frac{A_{n,n-1}B_n}{\rho_{n-1}(s)} \frac{\nabla}{\nabla x_n(s)}\rho_n(s),$$

as well as the identities $\rho_n(s) = \rho_{n-1}(s+1)\sigma(s+1)$, $x_n(s) = x_{n-1}(s + \frac{1}{2})$ and the Pearson equation (8) for $\rho_{n-1}(s)$, we find

$$\Delta^{(n-1)}P_n(x(s))_q = A_{n,n-1}B_n\tau_{n-1}(s).$$

Thus

$$a_n = \frac{A_{n,n-1}B_n\tilde{\tau}'_{n-1}}{[n]_q!} = B_n \prod_{k=0}^{n-1} \left\{ \alpha_q(n+k-1)\tilde{\tau}' + [n+k-1]_q \frac{\tilde{\sigma}''}{2} \right\},$$

and

$$\frac{b_n}{a_n} = \frac{[n]_q\tilde{\tau}_{n-1}(0)}{\tilde{\tau}'_{n-1}} + c_3([n]_q - n).$$

So

$$\alpha_n = \frac{B_n}{B_{n+1}} \frac{\alpha_q(n-1)\tilde{\tau}' + [n-1]_q \frac{\tilde{\sigma}''}{2}}{(\alpha_q(2n-1)\tilde{\tau}' + [2n-1]_q \frac{\tilde{\sigma}''}{2})(\alpha_q(2n)\tilde{\tau}' + [2n]_q \frac{\tilde{\sigma}''}{2})} = -\frac{B_n}{B_{n+1}} \frac{\lambda_n}{[n]_q} \frac{[2n]_q[2n+1]_q}{\lambda_{2n}\lambda_{2n+1}}$$

and

$$\beta_n = \frac{[n]_q\tilde{\tau}_{n-1}(0)}{\tilde{\tau}'_{n-1}} - \frac{[n+1]_q\tilde{\tau}_n(0)}{\tilde{\tau}'_n} + c_3([n]_q + 1 - [n+1]_q).$$

Using the Rodrigues formula the following difference-recurrent relation follows [1, 23]

$$\sigma(s) \frac{\nabla P_n(x(s))_q}{\nabla x(s)} = \frac{\lambda_n}{[n]_q\tau'_n} \left[\tau_n(s)P_n(x(s))_q - \frac{B_n}{B_{n+1}}P_{n+1}(x(s))_q \right],$$

where $\tau_n(s)$ is given by (6), where the identity $\tilde{\tau}'_n = -\frac{\lambda_{2n+1}}{[2n+1]_q}$ has been used.

Then, using the explicit expression for the coefficient α_n , we find

$$\sigma(s) \frac{\nabla P_n(x(s))_q}{\nabla x(s)} = \frac{\lambda_n}{[n]_q} \frac{\tau_n(s)}{\tau'_n} P_n(x(s))_q - \frac{\alpha_n\lambda_{2n}}{[2n]_q} P_{n+1}(x(s))_q. \quad (20)$$

This equation defines a raising operator in terms of the backward difference in the sense that we can obtain the polynomial P_{n+1} of degree $n + 1$ from the lower degree polynomial P_n .

From the above equation and using the identity $\nabla = \Delta - \nabla\Delta$, the second order difference equation and the three terms recurrence relation we find [1] lowering-type operator:

$$\begin{aligned} [\sigma(s) + \tau(s)\Delta x(s - \frac{1}{2})] \frac{\Delta P_n(x(s))_q}{\Delta x(s)} &= \frac{\gamma_n\lambda_{2n}}{[2n]_q} P_{n-1}(x(s))_q + \\ \left[\frac{\lambda_n}{[n]_q} \frac{\tau_n(s)}{\tau'_n} - \lambda_n\Delta x(s - \frac{1}{2}) - \frac{\lambda_{2n}}{[2n]_q}(x(s) - \beta_n) \right] &P_n(x(s))_q. \end{aligned} \quad (21)$$

The most general polynomial solution of the q -hypergeometric equation (1) corresponds to the case

$$\sigma(s) = A \prod_{i=1}^4 [s - s_i]_q = C q^{-2s} \prod_{i=1}^4 (q^s - q^{s_i}), \quad A, C, \text{ not vanishing constants} \quad (22)$$

and has the form [24]

$$P_n(s)_q = D_n {}_4\phi_3 \left(\begin{matrix} q^{-n}, q^{2\mu+n-1+\sum_{i=1}^4 s_i}, q^{s_1-s}, q^{s_1+s+\mu} \\ q^{s_1+s_2+\mu}, q^{s_1+s_3+\mu}, q^{s_1+s_4+\mu} \end{matrix}; q, q \right), \quad (23)$$

where D_n is a normalizing constant and the basic hypergeometric series ${}_p\phi_q$ are defined by [17]

$${}_r\phi_p \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_p \end{matrix}; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1; q)_k \cdots (a_r; q)_k}{(b_1; q)_k \cdots (b_p; q)_k} \frac{z^k}{(q; q)_k} \left[(-1)^k q^{\frac{k}{2}(k-1)} \right]^{p-r+1},$$

and

$$(a; q)_k = \prod_{m=0}^{k-1} (1 - aq^m), \quad (24)$$

is the q -analogue of the Pochhammer symbol. Instances of such polynomials are the Askey–Wilson polynomials, the q -Racah polynomials and big q -Jacobi polynomials among others [17, 24].

3 The orthonormal functions on nonuniform lattices

In this section we will introduce a set of orthonormal functions which are orthogonal with respect to the unit weight [6, 27]

$$\varphi_n(s) = \sqrt{\rho(s)/d_n^2} P_n(x(s))_q, \quad (25)$$

e.g. for the case of discrete orthogonality we have

$$\sum_{s_i=a}^{b-1} \varphi_n(s_i) \varphi_m(s_i) \Delta x(s_i - \frac{1}{2}) = \delta_{nm}.$$

Next, we will establish several important properties of such functions which generalize, to the nonuniform lattices, the ones given in [18]. In the following we will use the notation $\Theta(s) = \sigma(s) + \tau(s) \Delta x(s - \frac{1}{2})$.

First of all, inserting (25) into (1), (17), (20), (21) we obtain that they satisfy the following difference equation:

$$\begin{aligned} & \sqrt{\Theta(s)\sigma(s+1)} \frac{1}{\Delta x(s)} \varphi_n(s+1) + \sqrt{\Theta(s-1)\sigma(s)} \frac{1}{\nabla x(s)} \varphi_n(s-1) - \\ & \left(\frac{\Theta(s)}{\Delta x(s)} + \frac{\sigma(s)}{\nabla x(s)} \right) \varphi_n(s) + \lambda_n \Delta x(s - \frac{1}{2}) \varphi_n(s) = 0, \end{aligned} \quad (26)$$

the three term recurrence relation:

$$\alpha_n \frac{d_{n+1}}{d_n} \varphi_{n+1}(s) + \gamma_n \frac{d_{n-1}}{d_n} \varphi_{n-1}(s) + (\beta_n - x(s)) \varphi_n(s) = 0, \quad (27)$$

the raising-type formula:

$$L^+(s, n)\varphi_n(s) = \alpha_n \frac{\lambda_{2n}}{[2n]_q} \frac{d_{n+1}}{d_n} \varphi_{n+1}(s), \quad (28)$$

and the lowering-type formula:

$$L^-(s, n)\varphi_n(s) = \gamma_n \frac{\lambda_{2n}}{[2n]_q} \frac{d_{n-1}}{d_n} \varphi_{n-1}(s), \quad (29)$$

where the raising-type operator $L^+(s, n)$ and the lowering-type operator $L^-(s, n)$ are given by

$$L^+(s, n) \equiv \left[\frac{\lambda_n}{[n]_q} \frac{\tau_n(s)}{\tau'_n} - \frac{\sigma(s)}{\nabla x(s)} \right] I + \sqrt{\Theta(s-1)\sigma(s)} \frac{1}{\nabla x(s)} E^-, \quad (30)$$

and

$$L^-(s, n) \equiv \left[-\frac{\lambda_n}{[n]_q} \frac{\tau_n(s)}{\tau'_n} + \lambda_n \Delta x(s - \frac{1}{2}) + \frac{\lambda_{2n}}{[2n]_q} (x(s) - \beta_n) - \frac{\Theta(s)}{\Delta x(s)} \right] I + \sqrt{\Theta(s)\sigma(s+1)} \frac{1}{\Delta x(s)} E^+, \quad (31)$$

respectively. In the above formulas $E^- f(s) = f(s-1)$, $E^+ f(s) = f(s+1)$ and I is the identity operator.

Notice that the last two formulas have a remarkable property of giving all the solutions $\varphi_n(s)$. In fact, from (31) setting $n=0$ and taking into account that $\varphi_{-1}(s) \equiv 0$ we can obtain $\varphi_0(s)$. Then, substituting the obtained function in (30), we can find all the functions $\varphi_1(s), \dots, \varphi_n(s), \dots$.

Proposition 3.1 *The raising and lowering operators (30) and (31) are mutually adjoint.*

Proof: The proof is straightforward. In fact using the boundary condition and after some calculations we obtain, in the case of discrete orthogonality, the expression

$$\begin{aligned} & \sum_{s_i=a}^{b-1} \varphi_{n+1}(s_i) \left[\frac{[2n]_q}{\lambda_{2n}} L^+(s_i, n)\varphi_n(s_i) \right] \Delta x(s_i - \frac{1}{2}) \\ &= \sum_{s_i=a}^{b-1} \left[\frac{[2n+2]_q}{\lambda_{2n+2}} L^-(s_i, n+1)\varphi_{n+1}(s_i) \right] \varphi_n(s_i) \Delta x(s_i - \frac{1}{2}) = \alpha_n \frac{d_{n+1}}{d_n}. \end{aligned}$$

The other cases can be done in an analogous way. ■

Proposition 3.2 *The operator corresponding to the eigenvalue λ_n in (26) is self adjoint.*

Proof: Again we will prove the result in the case of discrete orthogonality. Using the orthogonality conditions $\sigma(a)\rho(a) = \sigma(b)\rho(b) = 0$ (which is a consequence of (13)), we can write

$$\begin{aligned} & \sum_{s_i=a}^{b-1} \varphi_n(s_i) \sqrt{\Theta(s_i-1)\sigma(s_i)} \frac{1}{\nabla x(s_i)} \varphi_l(s_i-1) \Delta x(s_i - \frac{1}{2}) \\ &= \sum_{s'_i=a-1}^{b-2} \varphi_n(s'_i+1) \sqrt{\Theta(s'_i)\sigma(s'_i+1)} \frac{1}{\nabla x(s'_i+1)} \varphi_l(s'_i) \Delta x(s'_i + \frac{1}{2}) \end{aligned}$$

$$\begin{aligned}
&= \sum_{s_i=a}^{b-1} \varphi_n(s_i+1) \sqrt{\Theta(s_i)\sigma(s_i+1)} \frac{1}{\nabla x(s_i+1)} \varphi_l(s_i) \Delta x(s_i+\frac{1}{2}) + \\
&\quad \varphi_n(a) \sqrt{\Theta(a-1)\sigma(a)} \frac{1}{\nabla x(a)} \varphi_l(a-1) \Delta x(a-\frac{1}{2}) - \\
&\quad \varphi_n(b) \sqrt{\Theta(b-1)\sigma(b)} \frac{1}{\nabla x(b)} \varphi_l(b-1) \Delta x(b-\frac{1}{2}),
\end{aligned}$$

where in the last two sums we first take the operations Δ and ∇ , and then substitute the corresponding value: e.g. $\Delta x(a) = x(a+1) - x(a)$.

Now, we use the fact that $\varphi_n(s) = \sqrt{\rho(s)/d_n^2} P_n(x(s))_q$, as well as the boundary conditions $\sigma(a)\rho(a) = \sigma(b)\rho(b) = 0$, so

$$\sqrt{\Theta(a-1)\sigma(a)} \varphi_n(a) \varphi_l(a-1) = \sqrt{\Theta(b-1)\sigma(b)} \varphi_n(b) \varphi_l(b-1) = 0.$$

The other terms can be transformed in a similar way. All these yield the expression

$$\begin{aligned}
&\sum_{s_i=a}^{b-1} \varphi_l(s_i) \left\{ \sqrt{\Theta(s_i)\sigma(s_i+1)} \frac{1}{\Delta x(s_i)} \varphi_n(s_i+1) \Delta x(s_i+\frac{1}{2}) + \right. \\
&\quad \left. \sqrt{\Theta(s_i-1)\sigma(s_i)} \frac{1}{\nabla x(s_i)} \varphi_n(s_i-1) \Delta x(s_i-\frac{1}{2}) \right\} = \\
&= \sum_{s_i=a}^{b-1} \varphi_n(s_i) \left\{ \sqrt{\Theta(s_i)\sigma(s_i+1)} \frac{1}{\Delta x(s_i)} \varphi_l(s_i+1) \Delta x(s_i+\frac{1}{2}) + \right. \\
&\quad \left. \sqrt{\Theta(s_i-1)\sigma(s_i)} \frac{1}{\nabla x(s_i)} \varphi_l(s_i-1) \Delta x(s_i-\frac{1}{2}) \right\},
\end{aligned}$$

from where the proposition easily follows. ■

4 Factorization of difference equation of hypergeometric type on the nonuniform lattice

We will define from (26) the following operator

$$\begin{aligned}
H(s, n) &\equiv \sqrt{\Theta(s-1)\sigma(s)} \frac{1}{\nabla x(s)} E^- + \sqrt{\Theta(s)\sigma(s+1)} \frac{1}{\Delta x(s)} E^+ - \\
&\quad \left(\frac{\Theta(s)}{\Delta x(s)} + \frac{\sigma(s)}{\nabla x(s)} - \lambda_n \Delta x(s-\frac{1}{2}) \right) I.
\end{aligned}$$

Clearly, the orthonormal functions satisfy

$$H(s, n) \varphi_n(s) = 0.$$

Let us rewrite the raising and lowering operators in the following way

$$\begin{aligned}
L^+(s, n) &= u(s, n) I + \sqrt{\Theta(s-1)\sigma(s)} \frac{1}{\nabla x(s)} E^-, \\
L^-(s, n) &= v(s, n) I + \sqrt{\Theta(s)\sigma(s+1)} \frac{1}{\Delta x(s)} E^+,
\end{aligned}$$

where, as before, $\Theta(s) = \sigma(s) + \tau(s)\Delta x(s - \frac{1}{2})$, and

$$u(s, n) = \frac{\lambda_n}{[n]_q} \frac{\tau_n(s)}{\tau'_n} - \frac{\sigma(s)}{\nabla x(s)},$$

$$v(s, n) = -\frac{\lambda_n}{[n]_q} \frac{\tau_n(s)}{\tau'_n} + \lambda_n \Delta x(s - \frac{1}{2}) + \frac{\lambda_{2n}}{[2n]_q} (x(s) - \beta_n) - \frac{\Theta(s)}{\Delta x(s)}.$$

Proposition 4.1 *The functions $u(s, n)$ and $v(s, n)$ satisfy $u(s+1, n) = v(s, n+1)$, or, equivalently $u(s+1, n-1) = v(s, n)$.*

The proof of the above proposition is straightforward but cumbersome. We will include it in appendix A. If we now calculate

$$L^-(s, n+1)L^+(s, n) = v(s, n+1)u(s, n) + \Theta(s)\sigma(s+1) \left(\frac{1}{\Delta x(s)} \right)^2 +$$

$$+ u(s+1, n) \left\{ \sqrt{\Theta(s-1)\sigma(s)} \frac{1}{\nabla x(s)} E^- + \sqrt{\Theta(s)\sigma(s+1)} \frac{1}{\Delta x(s)} E^+ \right\},$$

and substitute the values for $u(s, n)$, $v(s, n)$ and $H(s, n)$ we get

$$L^-(s, n+1)L^+(s, n) = h^\mp(n)I + u(s+1, n)H(s, n),$$

where the function

$$h^\mp(n) = \left(\frac{\lambda_n}{[n]_q} \frac{\tau_n(s+1)}{\tau'_n} - \frac{\sigma(s+1)}{\nabla x(s+1)} \right) \left(\frac{\lambda_n}{[n]_q} \frac{\tau_n(s)}{\tau'_n} - \lambda_n \Delta x(s - \frac{1}{2}) \right) +$$

$$\frac{\lambda_n}{[n]_q} \frac{\tau_n(s+1)}{\tau'_n} \frac{\Theta(s)}{\Delta x(s)},$$

is independent of s . In fact, applying the last equality to the orthonormal function $\varphi_n(s)$ and taking into account (28) and (29),

$$h^\mp(n) = \frac{\lambda_{2n}}{[2n]_q} \frac{\lambda_{2n+2}}{[2n+2]_q} \alpha_n \gamma_{n+1}.$$

Similarly,

$$L^+(s, n-1)L^-(s, n) = h^\pm(n)I + u(s, n-1)H(s, n),$$

where

$$h^\pm(n) = \left(-\frac{\lambda_n}{[n]_q} \frac{\tau_n(s-1)}{\tau'_n} + \frac{\lambda_{2n}}{[2n]_q} (x(s-1) - \beta_n) + \lambda_n \Delta x(s - \frac{3}{2}) \right) \times$$

$$\left(-\frac{\lambda_n}{[n]_q} \frac{\tau_n(s)}{\tau'_n} + \frac{\lambda_{2n}}{[2n]_q} (x(s) - \beta_n) + \frac{\sigma(s)}{\nabla x(s)} \right) -$$

$$\left(-\frac{\lambda_n}{[n]_q} \frac{\tau_n(s)}{\tau'_n} + \frac{\lambda_{2n}}{[2n]_q} (x(s) - \beta_n) \right) \left(\frac{\Theta(s-1)}{\Delta x(s-1)} \right),$$

is independent of s . Furthermore, applying the last expression to the functions $\varphi_n(s)$, and taking into account (28) and (29), we obtain

$$h^\pm(n) = \frac{\lambda_{2n-2}}{[2n-2]_q} \frac{\lambda_{2n}}{[2n]_q} \alpha_{n-1} \gamma_n.$$

Remark: Notice that $h^\pm(n+1) = h^\mp(n)$.

All the above results lead us to our main theorem:

Theorem 4.1 *The operator $H(s, n)$, corresponding to the hypergeometric difference equation for orthonormal functions $\varphi_n(s)$, admits the following factorization –usually called the Infeld-Hull-type factorization–*

$$u(s+1, n)H(s, n) = L^-(s, n+1)L^+(s, n) - h^\mp(n)I, \quad (32)$$

and

$$u(s, n)H(s, n+1) = L^+(s, n)L^-(s, n+1) - h^\mp(n)I, \quad (33)$$

respectively.

Remark: Substituting in the above formulas the expression $x(s) = s$ we obtain the corresponding results for the uniform lattice cases (Hahn, Kravchuk, Meixner and Charlier), considered before by several authors, see e.g. [6, 18, 27] and by taking appropriate limits (see e.g. [17, 23]), we can recover the classical continuous case (Jacobi, Laguerre and Hermite).

5 Applications to some q -normalized orthogonal functions

For the sake of completeness we will apply the above results to several families of orthogonal q -polynomials and their corresponding orthonormal q -functions that are of interest and appear in several branches of mathematical physics. They are the Al-Salam & Carlitz polynomials I and II, the big q -Jacobi polynomials, the dual q -Hahn polynomials, the continuous q -Hermite and the celebrated q -Askey–Wilson polynomials.

The main data for these polynomials are taken from the nice survey [17] except the case of dual q -Hahn polynomials [3]. Nevertheless, they can be obtained also from the general formulas given in Section 2.

Finally, let us point out that similar factorization formulas were obtained by other authors, e.g. Miller in [21] considered the polynomials on the linear exponential lattice and Bangerezako studied the Askey–Wilson case. Our main aim in this section is to show how our general formulas lead, in a very easy way, to the needed factorization formulas of several families for normalized functions –not polynomials–.

5.1 The Al-Salam & Carlitz functions I and II

The Al-Salam & Carlitz polynomials I (and II) appear in certain models of q -harmonic oscillator, see e.g. [5, 8, 9, 22]. They are polynomials on the exponential lattice $x(s) = q^s \equiv x$, defined [17] by

$$U_n^{(a)}(x; q) = (-a)^n q^{\binom{n}{2}} {}_2\phi_1 \left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix} \middle| q; \frac{qx}{a} \right),$$

and constitute an orthogonal family with the orthogonality relation (15)

$$\int_a^1 U_n^{(a)}(x; q) U_m^{(a)}(x; q) \omega(x) d_q x = d_n^2 \delta_{nm},$$

where

$$\omega(x) = (qx, a^{-1}qx; q)_\infty, \quad \text{and} \quad d_n^2 = (-a)^n (1-q)(q; q)_n (q, a, a^{-1}q; q)_\infty q^{\binom{n}{2}}.$$

As usual, $(a_1, \dots, a_p; q)_n = (a_1; q)_n \cdots (a_p; q)_n$, and $(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k)$.

They satisfy a difference equation of the form (1) where

$$\sigma(x) = (x-1)(x-a), \quad \tau(x) = \tilde{\tau}(x) = \tau'x + \tau(0), \quad \text{being } \tau' = \frac{q^{1/2}}{1-q}, \quad \tau(0) = q^{1/2} \frac{1+a}{q-1}.$$

The eigenvalues λ_n and the coefficients of the TTRR are given by

$$\lambda_n = [n]_q \frac{q^{1-n/2}}{q-1} \quad \text{and} \quad \alpha_n = 1, \quad \beta_n = (1+a)q^n, \quad \gamma_n = aq^{n-1}(q^n - 1),$$

respectively. In this case we have

$$\tilde{\sigma}'' = 1, \quad \tilde{\sigma}'(0) = -\frac{a+1}{2}, \quad \tilde{\sigma}(0) = a, \quad \tau'_n = \frac{q^{\frac{1}{2}-n}}{1-q}, \quad \tau_n(0) = q^{\frac{1-n}{2}} \frac{a+1}{q-1}.$$

The corresponding normalized functions (25) are

$$\varphi_n(x) = \sqrt{\frac{(qx, a^{-1}qx; q)_\infty (-a)^n q^{\binom{n}{2}}}{(1-q)(q; q)_n (q, a, q/a; q)_\infty}} {}_2\phi_1 \left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix} \middle| q; \frac{qx}{a} \right).$$

Defining now the Hamiltonian for these functions $\varphi_n(x)$

$$H(x, n) = \frac{\sqrt{a(x-1)(x-a)}}{x(1-q^{-1})} E^- + \frac{\sqrt{a(qx-1)(qx-a)}}{x(q-1)} E^+ + \left(\frac{q^{1-n}}{1-q} x + \frac{q(a+1)}{q-1} - \frac{[2]_q}{k_q} x^{-1} \right) I,$$

and using that $u(x, n) = \frac{aq}{1-q} x^{-1}$, $v(x, n) = u(qx, n-1) = \frac{a}{1-q} x^{-1}$, thus

$$L^+(x, n) = u(x, n)I + q \frac{\sqrt{a(x-1)(x-a)}}{x(q-1)} E^-, \quad \text{where } E^- f(x) = f(q^{-1}x),$$

and

$$L^-(x, n) = v(x, n)I + \frac{\sqrt{a(qx-1)(qx-a)}}{x(q-1)} E^+, \quad \text{where } E^+ f(x) = f(qx),$$

we have

$$L^-(x, n+1)L^+(x, n) = \frac{aq^{1-n}(q^{n+1}-1)}{(q-1)^2} I + v(x, n+1)H(x, n),$$

and

$$L^+(x, n-1)L^-(x, n) = \frac{aq^{2-n}(q^n-1)}{(q-1)^2} I + u(x, n-1)H(x, n),$$

which give the factorization formulas for the Al-Salam & Carlitz functions I. If we now taking into account that (see [17, p. 115])

$$V_n^{(a)}(x; q) = U_n^{(a)}(x; q^{-1}),$$

then, the factorization for the Al-Salam & Carlitz functions II

$$\varphi_n(s) = q^{\binom{s}{2}} \sqrt{\frac{a^{s+n}(aq; q)_\infty q^{\binom{n+1}{2}}}{(q, aq; q)_s (1-q)(q; q)_n}} {}_2\phi_0 \left(\begin{matrix} q^{-n}, x \\ - \end{matrix} \middle| q; \frac{q^n}{a} \right),$$

follows from the factorization for the Al-Salam & Carlitz functions I simply by changing q to q^{-1} .

5.2 The big q -Jacobi functions

Now we will consider the most general family of q -polynomials on the exponential lattice, the so-called big q -Jacobi polynomials, that appear in the representation theory of the quantum algebras [31]. They were introduced by Hahn in 1949 and are defined [17] by

$$P_n(x; a, b, c; q) = \frac{(aq; q)_n (cq; q)_n}{(abq^{n+1}; q)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix} \middle| q; q \right), \quad x(s) = q^s \equiv x.$$

They constitute an orthogonal family, i.e.,

$$\int_{cq}^{aq} \omega(x) P_n(x; a, b, c; q) P_m(x; a, b, c; q) d_q x = d_n^2 \delta_{nm}$$

where

$$\omega(x) = \frac{(a^{-1}x; q)_\infty (c^{-1}x; q)_\infty}{(x; q)_\infty (bc^{-1}x; q)_\infty},$$

$$d_n^2 = \frac{aq(1-q)(q, c/a, aq/c, abq^2; q)_\infty (1-abq)(q, bq, abq/c; q)_n (-ac)^{-n} q^{-\binom{n}{2}}}{(aq, bq, cq, abq/c; q)_\infty (abq, abq^{n+1}, abq^{n+1})_n}.$$

They satisfy the difference equation (1) with

$$\sigma(x) = q^{-1}(x-aq)(x-cq) \quad \text{and} \quad \tau(x) = \tilde{\tau}(x) = \tau'x + \tau(0),$$

where

$$\tau' = \frac{1-abq^2}{(1-q)q^{1/2}} \quad \text{and} \quad \tau(0) = q^{1/2} \frac{a(bq-1) + c(aq-1)}{1-q},$$

and

$$\lambda_n = -q^{-n/2} [n]_q \frac{1-abq^{n+1}}{1-q}.$$

They satisfy a TTRR, whose coefficients are

$$\alpha_n = 1, \quad \beta_n = 1 - A_n - C_n, \quad \gamma_n = C_n A_{n-1},$$

where

$$A_n = \frac{(1-aq^{n+1})(1-cq^{n+1})(1-abq^{n+1})}{(1-abq^{2n+1})(1-abq^{2n+2})},$$

$$C_n = -acq^{n+1} \frac{(1-q^n)(1-bq^n)(1-abc^{-1}q^n)}{(1-abq^{2n})(1-abq^{2n+1})}.$$

Also, we have

$$\begin{aligned} \tilde{\sigma}'' &= \frac{1+abq^2}{q}, \quad \tilde{\sigma}'(0) = -\frac{abq+acq+a+c}{2}, \quad \tilde{\sigma}(0) = acq, \\ \tau'_n &= \frac{q^{-n}-abq^{n+2}}{q^{1/2}(1-q)}, \quad \tau_n(0) = q^{\frac{1-n}{2}} \frac{a(bq^{1+n}-1) + c(aq^{1+n}-1)}{1-q}. \end{aligned}$$

The normalized big q -Jacobi functions are defined by

$$\varphi_n(s) = \sqrt{\frac{(x/a, x/c; q)_\infty (aq, bq, abq/c; q)_\infty (abq, aq, aq, cq, cq; q)_n (-ac)^n}{(x, bx/c, c/a, aq/c, abq^2; q)_\infty (1-q)aq(1-abq)(q, bq, abq/c; q)_n}} \times {}_3\phi_2 \left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix} \middle| q; q \right).$$

The corresponding Hamiltonian is

$$H(x, n) = \frac{\sqrt{a(x-q)(x-aq)(x-cq)(bx-cq)}}{x(q-1)} E^- + q \frac{\sqrt{a(x-1)(x-a)(x-c)(bx-c)}}{x(q-1)} E^+ + \left(\frac{1+abq^{2n+1}}{q^n(1-q)} x - \frac{q(a+ab+c+ac)}{1-q} + \frac{acq(q+1)}{1-q} x^{-1} \right) I.$$

Furthermore,

$$u(x, n) = \frac{abq^{n+1}}{1-q} x + D_n - \frac{acq^2}{q-1} x^{-1} \quad \text{and} \quad v(x, n) = \frac{abq^{n+1}}{1-q} x + D_{n-1} - \frac{acq}{q-1} x^{-1},$$

where

$$D_n = \frac{ab(ab+ac+a+c)q^{2n+3} - a(b+c+ab+bc)q^{n+2}}{(1-abq^{2n+2})(1-q)},$$

thus

$$L^+(x, n) = u(x, n)I + \frac{\sqrt{a(x-q)(x-aq)(x-cq)(bx-cq)}}{x(q-1)} E^-, \quad \text{where} \quad E^- f(x) = f(q^{-1}x),$$

and

$$L^-(x, n) = v(x, n)I + q \frac{\sqrt{a(x-1)(x-a)(x-c)(bx-c)}}{x(q-1)} E^+, \quad \text{where} \quad E^+ f(x) = f(qx),$$

so

$$L^-(x, n+1)L^+(x, n) = \delta_{n+1}\gamma_{n+1}I + v(x, n+1)H(x, n), \\ L^+(x, n-1)L^-(x, n) = \delta_n\gamma_n I + u(x, n-1)H(x, n),$$

where

$$\delta_n = \frac{(1-abq^{2n-1})(1-abq^{2n+1})}{q^{2n-1}(q-1)^2}.$$

The above formulas are the factorization formulas for the family of the big q -Jacobi normalized functions.

Since all discrete q -polynomials on the exponential lattice $x(s) = c_1q^s + c_3$ —the so called, q -Hahn class— can be obtained from the big q -Jacobi polynomials by a certain limit process (see e.g. [2, 17], then from the above formulas we can obtain the factorization formulas for the all other cases in the q -Hahn tableau. Of special interest are the q -Hahn polynomials and the big q -Laguerre polynomials, which are particular cases of the big q -Jacobi polynomials when $c = q^{-N-1}$, $N = 1, 2, \dots$, and $c = 0$, respectively.

5.3 The q -dual-Hahn functions

In this section we will deal with the q -dual-Hahn polynomials, introduced in [3, 24] and closely related with the Clebsh-Gordon coefficients of the q -algebras $SU_q(2)$ and $SU_q(1, 1)$ [3]. They are defined on the lattice $x(s) = [s]_q[s+1]_q$ by

$$W_n^c(x(s); a, b)_q = \frac{(-1)^n (q^{a-b+1}; q)_n (q^{a+c+1}; q)_n}{q^{n/2(3a-b+c+1+n)} k_q^n(q; q)_n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, q^{a-s}, q^{a+s+1} \\ q^{a-b+1}, q^{a+c+1} \end{matrix} \middle| q; q \right),$$

and satisfy a discrete orthogonality (14) with respect to the weight function

$$\rho(s) = \frac{q^{\frac{1}{2}((b-1)^2 - (2s-1)(a+c))} (q^{s-a+1}, q^{s-c+1}, q^{s+b+1}, q^{b-s}; q)_\infty}{(1-q)^{2(a+c-b)+1} (q, q, q^{s+a+1}, q^{s+c+1}; q)_\infty},$$

where $-\frac{1}{2} \leq a < b-1$, $|c| < a+1$, and for this weight function the norm is

$$d_n^2 = \frac{q^{\frac{1}{4}(-4ab-4bc+6a+6c-8b+6+4n(a+c-2b)-n^2+17n+2b^2)} (q^{b-c-n}, q^{b-a-n}; q)_\infty}{(1-q)^{2(a+c-b+1)+3n} [n]_q! (q, q^{a+c+n+1}; q)_\infty}.$$

These polynomials satisfy a TTRR (17) with

$$\alpha_n = 1,$$

$$\beta_n = q^{\frac{1}{2}(2n-b+c+1)} [b-a-n+1]_q [a+c+n+1]_q + q^{\frac{1}{2}(2n+2a+c-b+1)} [n]_q [b-c-n]_q + [a]_q [a+1]_q,$$

$$\gamma_n = q^{2n+c+a-b} [a+c+n]_q [b-a-n]_q [b-c-n]_q [n]_q,$$

and the second order difference equation (1), whose eigenvalues are $\lambda_n = [n]_q q^{\frac{1}{2}-\frac{n}{2}}$ and

$$\sigma(s) = q^{\frac{1}{2}(s+c+a-b+2)} [s-a]_q [s+b]_q [s-c]_q \quad \text{and} \quad \tau(x) = \tilde{\tau}(x) = \tau'x + \tau(0),$$

with $\tau' = -1$ and $\tau(0) = q^{\frac{1}{2}(a-b+c+1)} [a+1]_q [b-c-1]_q + q^{\frac{1}{2}(c-b+1)} [b]_q [c]_q$.

Also we will need the values

$$\tilde{\sigma}'' = k_q, \quad \tilde{\sigma}'(0) = \frac{1}{2k_q} (2[2]_q - q^{\frac{1}{2}-b} - q^{\frac{1}{2}+a} - q^{\frac{3}{2}+a+c-b} - q^{\frac{1}{2}+c}),$$

$$\tilde{\sigma}(0) = \frac{1}{2k_q^3} (2q^{1+a-b} + q^{-1} + q + 2q^{1+c-b} + 2q^{1+a+c} - (1+q)(q^{-b} + q^a + q^c + q^{1+a+c-b})),$$

$$\tau'_n = -q^{-n}, \quad \tau_n(0) = q^{\frac{1}{2}(c-b-n+1)} [c+\frac{n}{2}]_q [b-\frac{n}{2}]_q + q^{\frac{1}{2}(a+c-b+1-\frac{n}{2})} [a+\frac{n}{2}+1]_q [b-c-n-1]_q,$$

In this case, the Hamiltonian, associated with the q -dual Hahn normalized functions $\sqrt{\rho(s)/d_n^2} W_n^c(x(s); a, b)_q$, is

$$\begin{aligned} H(s, n) &= q^{\frac{1}{2}(c+a-b+2)} \frac{\sqrt{([s+1]_q^2 - [a]_q^2)([b]_q^2 - [s+1]_q^2)([s+1]_q^2 - [c]_q^2)}}{[2s+2]_q} E^+ + \\ & q^{\frac{1}{2}(c+a-b+2)} \frac{\sqrt{([s]_q^2 - [a]_q^2)([b]_q^2 - [s]_q^2)([s]_q^2 - [c]_q^2)}}{[2s]_q} E^- - q^{\frac{1}{2}-\frac{n}{2}} [n]_q [2s+1]_q I + \\ & q^{\frac{1}{2}(c+a-b+2)} \left(\frac{[s-a]_q [s+b]_q + [s-c]_q}{[2s]_q} - \frac{[s+1-a]_q [s+1+b]_q [s+1-c]_q}{[2s+2]_q} \right) I, \end{aligned}$$

where $E^+ f(s) = f(s+1)$ and $E^- f(s) = f(s-1)$. Then, using that

$$u(s, n) = q^{\frac{1}{2}-\frac{n}{2}} x(s+n/2) - q^{\frac{1}{2}+\frac{n}{2}} (q^{\frac{1}{2}(c-b-n+1)} [c+\frac{n}{2}]_q [b-\frac{n}{2}]_q +$$

$$q^{\frac{1}{2}(a+c-b+1-\frac{n}{2})} [a+\frac{n}{2}+1]_q [b-c-n-1]_q) - q^{\frac{1}{2}(s+c+a-b+2)} \frac{[s-a]_q [s+b]_q [s-c]_q}{[2s]_q},$$

and taking into account that $v(s, n) = u(s+1, n-1)$, we find

$$L^+(s, n) = u(s, n)I + q^{\frac{1}{2}(c+a-b+2)} \frac{\sqrt{([s]_q^2 - [a]_q^2)([b]_q^2 - [s]_q^2)([s]_q^2 - [c]_q^2)}}{[2s]_q} E^-,$$

and

$$L^-(s, n) = v(s, n)I + q^{\frac{1}{2}(c+a-b+2)} \frac{\sqrt{([s+1]_q^2 - [a]_q^2)([b]_q^2 - [s+1]_q^2)([s+1]_q^2 - [c]_q^2)}}{[2s+2]_q} E^+.$$

Thus

$$L^-(s, n+1)L^+(s, n) = q^{-2n}\gamma_{n+1}I + v(s, n+1)H(s, n),$$

and

$$L^+(s, n-1)L^-(s, n) = q^{-2n+2}\gamma_n I + u(s, n-1)H(s, n),$$

are the factorization formulas for the q -dual Hahn normalized functions.

5.4 The Askey–Wilson functions

Finally we will consider the family of Askey–Wilson polynomials. They are polynomials on the lattice $x(s) = \frac{1}{2}(q^s + q^{-s}) \equiv x$, defined by [17]

$$p_n(x(s); a, b, c, d) = \frac{(ab; q)_n (ac; q)_n (ad; q)_n}{a^n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, q^{n-1}abcd, ae^{-i\theta}, ae^{i\theta} \\ ab, ac, ad \end{matrix} \middle| q; q \right),$$

i.e., they correspond to the general case (23) when $q^{s_1} = a$, $q^{s_2} = b$, $q^{s_3} = c$, $q^{s_4} = d$. Their orthogonality relation is of the form

$$\int_{-1}^1 \omega(x) p_n(x; a, b, c, d) p_m(x; a, b, c, d) \sqrt{1-x^2} \kappa_q dx = \delta_{nm} d_n^2, \quad q^s = e^{i\theta}, \quad x = \cos \theta,$$

where

$$\omega(x) = \frac{h(x, 1)h(x, -1)h(x, q^{\frac{1}{2}})h(x, -q^{\frac{1}{2}})}{2\pi\kappa_q(1-x^2)h(x, a)h(x, b)h(x, c)h(x, d)}, \quad h(x, \alpha) = \prod_{k=0}^{\infty} [1 - 2\alpha x q^k + \alpha^2 q^{2k}],$$

and the norm is given by

$$d_n^2 = \frac{(abcdq^{n-1}; q)_n (abcdq^{2n}; q)_\infty}{(q^{n+1}, abq^n, acq^n, adq^n, bcq^n, bdq^n, cdq^n; q)_\infty}.$$

The Askey–Wilson polynomials satisfy the difference equation (1) with

$$\sigma(s) = -q^{-2s+1/2}\kappa_q^2(q^s - a)(q^s - b)(q^s - c)(q^s - d), \quad \kappa_q = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})$$

and $\tau(x) = \tilde{\tau}(x) = \tau'x + \tau(0)$, where

$$\tau' = 4(q-1)(1-abcd), \quad \tau(0) = 2(1-q)(a+b+c+d - abc - abd - acd - bcd).$$

Furthermore, they satisfy the TTRR (17) with coefficients

$$\alpha_n = 1, \quad \beta_n = \frac{a + a^{-1} - (A_n + C_n)}{2}, \quad \gamma_n = \frac{C_n A_{n-1}}{4},$$

where A_n, C_n are defined by

$$A_n = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})},$$

$$C_n = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})},$$

and whose eigenvalues are $\lambda_n = 4q^{-n+1}(1 - q^n)(1 - abcdq^{n-1})$. In addition, we have

$$\begin{aligned}\tilde{\sigma}'' &= -4(q-1)^2(1+abcd)q^{-1/2}, \\ \tilde{\sigma}'(0) &= (q-1)^2(a+b+c+d+abc+abd+acd+bcd)q^{-1/2}, \\ \tilde{\sigma}(0) &= (q-1)^2(1-ab-ac-ad-bc-bd-cd+abcd)q^{-1/2}, \\ \tau_n' &= 4q^{-n}(q-1)(1-abcdq^{2n}), \\ \tau_n(0) &= 2(q-1)(-a-b-c-d+(abc+abd+acd+bcd)q^n)q^{-n/2}.\end{aligned}$$

Defining now the normalized functions (see (15)) $\sqrt{\omega(x)/d_n^2}p_n(x; a, b, c, d)$, the corresponding Hamiltonian $H(s, n)$ is

$$\begin{aligned}H(s, n) &= \frac{2q^{3/2}}{[2s-1]_q}G(s, a, b, c, d)E^- + \frac{2q^{3/2}}{[2s+1]_q}G(s+1, a, b, c, d)E^+ + \\ &2\left(q^{-2s+1/2}\frac{\prod_{i=1}^4(1-q^{s_i+s})}{[2s+1]_q} + q^{-2s+1/2}\frac{\prod_{i=1}^4(q^s - q^{s_i})}{[2s-1]_q} + \right. \\ &\left. q^{-n+1}\kappa_q^2(1-q^n)(1-abcdq^{n-1})[2s]_q\right)I\end{aligned}$$

where

$$G(s, a, b, c, d) = \sqrt{\prod_{i=1}^4(1 - 2q^{s_i}q^{-1/2}x(s-1/2) + q^{-1}q^{2s_i})},$$

We now define

$$u(s, n) = D_n x_n(s) + D_n E_n + q^{-2s+1/2} \frac{(q^s - a)(q^s - b)(q^s - c)(q^s - d)}{[2s-1]_q}$$

where

$$\begin{aligned}D_n &= -4q^{-n/2+1/2}(q-1)(1-abcdq^{n-1}). \\ E_n &= \frac{(-a-b-c-d+(abc+abd+acd+bcd)q^n)q^{n/2}}{2(1-abcdq^{2n})}.\end{aligned}$$

Taking into account that $v(s, n) = u(s+1, n-1)$, we find

$$\begin{aligned}L^+(s, n) &= u(s, n)I + \frac{2q^{3/2}}{[2s-1]_q}G(s, a, b, c, d)E^-, \\ L^-(s, n) &= v(s, n)I + \frac{2q^{3/2}}{[2s+1]_q}G(s+1, a, b, c, d)E^+, \end{aligned}$$

where $E^-f(s) = f(s-1)$ and $E^+f(s) = f(s+1)$. Thus,

$$L^-(s, n+1)L^+(s, n) = D_{2n}D_{2n+2}\gamma_{n+1}I + v(s, n+1)H(s, n),$$

and

$$L^+(s, n-1)L^-(s, n) = D_{2n-2}D_{2n}\gamma_n I + u(s, n-1)H(s, n),$$

which is the factorization formula for the Askey–Wilson functions.

To conclude this paper let us consider the special case of Askey–Wilson polynomials when $a = b = c = d = 0$, i.e., the continuous q -Hermite polynomials

$$H_n(x|q) = 2^{-n} e^{in\theta} {}_2\phi_0 \left(\begin{matrix} q^{-n}, 0 \\ - \end{matrix} \middle| q; q^n e^{-2i\theta} \right), \quad x = \cos \theta.$$

These polynomials are closely related with the q -harmonic oscillator model introduced by Biedenharn [13] and Macfarlane [19], as it was pointed out in [9], where the factorization for the continuous q -Hermite polynomials were considered first. If we substitute $a = b = c = d = 0$ in the above formulas, we obtain the factorization for the q -Hermite functions

$$\varphi_n(x) = \sqrt{\frac{h(x, 1)h(x, -1)h(x, q^{1/2})h(x, -q^{1/2})(q^{n+1}; q)_\infty}{2\pi\kappa_q(1-x^2)}} H_n(x|q).$$

In fact, since for continuous q -Hermite polynomials

$$\sigma(s) = -\kappa_q^2 q^{2s+1/2}, \quad \tau(s) = 4(q-1)x(s), \quad \lambda_n = 4q^{-n+1}(1-q^n),$$

and the coefficients for the three-term recurrence relation are $\alpha_n = 1$, $\beta_n = 0$, $\gamma_n = (1-q^n)/4$, then we obtain

$$H(s, n) = \frac{2q^{3/2}}{[2s-1]_q} E^- + \frac{2q^{3/2}}{[2s+1]_q} E^+ + 2 \left(\frac{q^{-2s+1/2}}{[2s+1]_q} + \frac{q^{2s+1/2}}{[2s-1]_q} - q^{-n+1} \kappa_q^2 (1-q^n) [2s]_q \right) I,$$

$$L^+(s, n) = \left(-4q^{-n/2+1/2}(q-1)x(s+n/2) + \frac{q^{2s+1/2}}{[2s-1]_q} \right) I + \frac{2q^{3/2}}{[2s-1]_q} E^-,$$

$$L^-(s, n) = \left(-4q^{-n/2+1}(q-1)x(s+n/2+1/2) + \frac{q^{2s+5/2}}{[2s+1]_q} \right) I + \frac{2q^{3/2}}{[2s+1]_q} E^-$$

and $h^\pm(n) = 4\kappa_q^2 q^{-2n+1}(1-q^n)$.

Acknowledgements

The authors thank N. Atakishiyev and Yu. F. Smirnov for interesting discussions and remarks that allowed us to improve this paper substantially, as well as the referees for their remarks. The work has been partially supported by the Ministerio de Ciencias y Tecnología of Spain under the grant BFM-2000-0206-C04-02, the Junta de Andalucía under grant FQM-262 and the European project INTAS-2000-272.

Appendix A

Here, for the sake of completeness, we will prove Proposition 4.1, by showing that $u(s+1, n) - v(s, n+1) = 0$. To do that, we start with computing the difference

$$\begin{aligned} u(s+1, n) - v(s, n+1) &= \frac{\lambda_n}{[n]_q} \frac{\tau_n(s+1)}{\tau'_n} - \frac{\Delta\sigma(s)}{\Delta x(s)} + \\ &\frac{\lambda_{n+1}}{[n+1]_q} \frac{\tau_{n+1}(s)}{\tau'_{n+1}} - \lambda_{n+1} \Delta x\left(s - \frac{1}{2}\right) - \frac{\lambda_{2n+2}}{[2n+2]_q} (x(s) - \beta_{n+1}) + \frac{\tau(s) \Delta x\left(s - \frac{1}{2}\right)}{\Delta x(s)}. \end{aligned}$$

Now we use the expansion $\tau_n(s+1) = \tau'_n x_n(s+1) + \tau_n(0)$. Since

$$\begin{aligned} \frac{\Delta(x^2(s))}{\Delta x(s)} &= \frac{x^2(s+1) - x^2(s)}{x(s+1) - x(s)} = x(s+1) + x(s) = C_1 q^s (q+1) + C_2 q^{-s} (q^{-1} + 1) + 2C_3 = \\ &(C_1 q^{s+\frac{1}{2}} + C_2 q^{-s-\frac{1}{2}})[2]_q + 2C_3 = [2]_q x_1(s) + (2 - [2]_q)C_3, \end{aligned}$$

$$\begin{aligned} x(s)\Delta x(s - \frac{1}{2}) &= x(s)(C_1 q^{s-\frac{1}{2}}(q-1) + C_2 q^{-s+\frac{1}{2}}(q^{-1}-1)) = x(s)(C_1 q^s - C_2 q^{-s})k_q = \\ &(C_1^2 q^{2s} - C_2^2 q^{-2s})k_q + C_3(C_1 q^s - C_2 q^{-s})k_q, \end{aligned}$$

where $k_q = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$,

$$\frac{\Delta}{\Delta x(s)} \left(x(s)\Delta x(s - \frac{1}{2}) \right) = \left(\frac{(C_1^2 q^{2s+1} + C_2^2 q^{-2s-1})[2]_q + C_3(C_1 q^{s+\frac{1}{2}} + C_2 q^{-s-\frac{1}{2}})}{C_1 q^{s+\frac{1}{2}} - C_2 q^{-s-\frac{1}{2}}} \right) k_q,$$

and

$$\frac{\Delta}{\Delta x(s)} \left(\Delta x(s - \frac{1}{2}) \right) = \frac{\Delta}{\Delta x(s)} \left((C_1 q^s - C_2 q^{-s})k_q \right) = \frac{C_1 q^{s+\frac{1}{2}} + C_2 q^{-s-\frac{1}{2}}}{C_1 q^{s+\frac{1}{2}} - C_2 q^{-s-\frac{1}{2}}} k_q.$$

Then

$$\begin{aligned} \frac{\Delta \sigma(s)}{\Delta x(s)} &= \frac{\Delta}{\Delta x(s)} \left(\tilde{\sigma}(s) - \frac{1}{2} \tilde{\tau}(s) \Delta x(s - \frac{1}{2}) \right) = \\ &= \frac{\Delta}{\Delta x(s)} \left(\frac{\tilde{\sigma}''}{2} x^2(s) + \tilde{\sigma}'(0)x(s) + \tilde{\sigma}(0) - \frac{1}{2} (\tau' x(s) + \tau(0)) \Delta x(s - \frac{1}{2}) \right) = \\ &\frac{\tilde{\sigma}''}{2} ([2]_q x_1(s) + (2 - [2]_q)C_3) + \tilde{\sigma}'(0) - \frac{1}{2} \tau(0) \left(\frac{C_1 q^{s+\frac{1}{2}} + C_2 q^{-s-\frac{1}{2}}}{C_1 q^{s+\frac{1}{2}} - C_2 q^{-s-\frac{1}{2}}} \right) k_q - \\ &\frac{1}{2} \tau' \left(\frac{[2]_q (C_1^2 q^{2s+1} + C_2^2 q^{-2s-1}) + C_3 (C_1 q^{s+\frac{1}{2}} + C_2 q^{-s-\frac{1}{2}})}{C_1 q^{s+\frac{1}{2}} - C_2 q^{-s-\frac{1}{2}}} \right) k_q. \end{aligned}$$

This yields for $u(s+1, n) - v(s, n+1)$ the expression

$$\begin{aligned} &= \left[\frac{\lambda_n}{[n]_q} x_n(s+1) + \frac{\lambda_n}{[n]_q} \frac{\tau_n(0)}{\tau'_n} \right] - \left[\frac{\tilde{\sigma}''}{2} [2]_q x_1(s) + \frac{C_3}{2} (2 - [2]_q) \tilde{\sigma}'' + \tilde{\sigma}'(0) - \right. \\ &\frac{\tilde{\tau}'}{2} \left(\frac{[2]_q (C_1^2 q^{2s+1} + C_2^2 q^{-2s-1})}{C_1 q^{s+\frac{1}{2}} - C_2 q^{-s-\frac{1}{2}}} + \frac{C_3 x_1(s) - C_3^2}{C_1 q^{s+\frac{1}{2}} - C_2 q^{-s-\frac{1}{2}}} \right) k_q - \\ &\frac{\tau(0)}{2} \left(\frac{x_1(s) - C_3}{C_1 q^{s+\frac{1}{2}} - C_2 q^{-s-\frac{1}{2}}} \right) k_q \left. \right] + \frac{\lambda_{n+1}}{[n+1]_q} \frac{\tau_{n+1}(s)}{\tau'_{n+1}} - \lambda_{n+1} \Delta x(s - \frac{1}{2}) - \\ &\frac{\lambda_{2n+2}}{[2n+2]_q} \left[C_1 q^s + C_2 q^{-s} + C_3 - \frac{[n+1]_q \tau_n(0)}{\tau'_n} + \right. \\ &\left. \frac{[n+2]_q \tau_{n+1}(0)}{\tau'_{n+1}} - C_3 (1 + [n+1]_q - [n+2]_q) \right] + \frac{\tau(s)\Delta x(s - \frac{1}{2})}{\Delta x(s)}. \end{aligned}$$

Next, we expand $\Delta x_n(s)$ and $\frac{\tilde{\sigma}''}{2}[2]_q x_1(s)$, make some straightforward calculations and use the identities:

$$\begin{aligned} \frac{\lambda_n}{[n]_q} \frac{\tau_n(0)}{\tau'_n} + [n+1]_q \frac{\lambda_{2n+2}}{[2n+2]_q} \frac{\tau_n(0)}{\tau'_n} &= \left(\frac{\lambda_n}{[n]_q} + [n+1]_q \frac{\lambda_{2n+2}}{[2n+2]_q} \right) \frac{\tau_n(0)}{\tau'_n} = -[n+2]_q \tau_n(0), \\ \frac{\lambda_{n+1}}{[n+1]_q} \frac{\tau_{n+1}(s)}{\tau'_{n+1}} - [n+2]_q \frac{\lambda_{2n+2}}{[2n+2]_q} \frac{\tau_{n+1}(0)}{\tau'_{n+1}} &= [n+1]_q \tau_{n+1}(0) + \frac{\lambda_{n+1}}{[n+1]_q} x_{n+1}(s), \end{aligned}$$

as well as

$$\begin{aligned} & \frac{\lambda_n}{[n]_q} (C_1 q^{s+1+\frac{n}{2}} + C_2 q^{-s-1-\frac{n}{2}}) - \frac{\tilde{\sigma}''}{2} [2]_q (C_1 q^{s+\frac{1}{2}} + C_2 q^{-s-\frac{1}{2}}) - \frac{\lambda_{2n+2}}{[2n+2]_q} (C_1 q^s + C_2 q^{-s}) - \\ & \lambda_{n+1} (C_1 q^s - C_2 q^{-s}) k_q + \frac{1}{2} \tau' (C_1 q^{s+\frac{1}{2}} + C_2 q^{-s-\frac{1}{2}}) (q + q^{-1}) = \frac{C_1 q^s \tau'}{2} (q^{n+\frac{1}{2}} + q^{\frac{1}{2}}) + \\ & \frac{C_1 q^s \tilde{\sigma}''}{2(q^{\frac{1}{2}} - q^{-\frac{1}{2}})} (q^{n+\frac{1}{2}} - q^{\frac{1}{2}}) + \frac{C_2 q^{-s} \tau'}{2} (q^{-n-\frac{1}{2}} + q^{-\frac{1}{2}}) + \frac{C_2 q^{-s} \tilde{\sigma}''}{2(q^{\frac{1}{2}} - q^{-\frac{1}{2}})} (-q^{-n-\frac{1}{2}} + q^{-\frac{1}{2}}) = \\ & = -\frac{\lambda_{n+1}}{[n+1]_q} (C_1 q^{s+\frac{n+1}{2}} + C_2 q^{-s-\frac{n+1}{2}}), \end{aligned}$$

we find

$$\begin{aligned} & = -\frac{\lambda_{n+1}}{[n+1]_q} (C_1 q^{s+\frac{n+1}{2}} + C_2 q^{-s-\frac{n+1}{2}}) + C_3 \frac{\lambda_n}{[n]_q} - [n+2]_q \tau_n(0) - C_3 \tilde{\sigma}'' - \tilde{\sigma}'(0) + \frac{1}{2} \tau' C_3 k_q + \\ & \frac{1}{2} \tau(0) k_q + [n+1]_q \tau_{n+1}(0) + \frac{\lambda_{n+1}}{[n+1]_q} (C_1 q^{s+\frac{n+1}{2}} + C_2 q^{-s-\frac{n+1}{2}}) + C_3 + \frac{\lambda_{2n+2}}{[2n+2]_q} C_3 ([n+1]_q - [n+2]_q). \end{aligned}$$

Finally, we substitute the expression for $\tau_n(0)$ and use the identities

$$\begin{aligned} & -[n+2]_q [n]_q - 1 + [n+1]_q [n+1]_q = 0, \\ & -[n+2]_q (q^{n/2} + q^{-n/2}) + k_q + [n+1]_q (q^{(n+1)/2} + q^{-(n+1)/2}) = 0, \end{aligned}$$

and the result follows.

References

- [1] R. Álvarez-Nodarse and J. Arvesú, On the q -polynomials on the exponential lattice $x(s) = c_1 q^s + c_3$. *Integral Trans. and Special Funct.* **8** (1999), 299-324.
- [2] R. Álvarez-Nodarse y J. C. Medem, q -Classical polynomials and the q -Askey and Nikiforov-Uvarov. *J. Comput. Appl. Math.* (2001), in press.
- [3] R. Álvarez-Nodarse and Yu.F. Smirnov, q -Dual Hahn polynomials on the non-uniform lattice $x(s) = [s]_q [s+1]_q$ and the q -algebras $SU_q(1,1)$ and $SU_q(2)$. *J. Phys. A.* **29** (1996), 1435-1451.
- [4] N. M. Atakishiyev, Construction of the dynamical symmetry group of the relativistic harmonic oscillator by the Infeld-Hull factorization method. In *Group Theoretical Methods in Physics*, M. Serdaroglu and E. Inonu. (Eds.). Lecture Notes in Physics, Springer-Verlag, **180**, 1983, 393-396. Ibid. *Theor. Math. Phys.* **56** (1984), 563-572.
- [5] R. Askey and S. K. Suslov, The q -harmonic oscillator and the Al-Salam and Carlitz polynomials. *Lett. Math. Phys.* **29** (1993), 123-132.
- [6] N. M. Atakishiyev and B. Wolf, Approximation on a finite set of points through Kravchuk functions. *Rev. Mex. Fis.* **40** (1994), 1055-1062.
- [7] N. M. Atakishiyev, M. Rahman, and S. K. Suslov, On Classical Orthogonal Polynomials. *Constructive Approx.* **11** (1995), 181-226.
- [8] N. M. Atakishiyev and S. K. Suslov, Difference analogs of the harmonic oscillator. *Theoret. and Math. Phys.* **85** (1991), 442-444.
- [9] N. M. Atakishiyev and S. K. Suslov, A realization of the q -harmonic oscillator. *Theoret. and Math. Phys.* **87** (1991), 1055-1062.

- [10] N. M. Atakishiyev, A. Frank, and K. B. Wolf, A simple difference realization of the Heisenberg q -algebra. *J. Math. Phys.* **35** (1994), 3253-3260.
- [11] G. Bangerezako, Discrete Darboux transformation for discrete polynomials of hypergeometric type. *J. Phys A: Math. Gen.* **31** (1998), 2191-2196.
- [12] G. Bangerezako, The factorization method for the Askey–Wilson polynomials. *J. Comput. Appl. Math.* **107** (1999), 219-232.
- [13] L. C. Biedenharn, The quantum group $SU_q(2)$ and a q -analogue of the boson operators. *J. Phys. A.* **22** (1989), L873-L878.
- [14] G. Darboux, *Théorie des Surfaces*. vol II (Paris, Gauthier-Villars).
- [15] G. Gasper and M. Rahman, *Basic Hypergeometric Series*. Cambridge University Press, 1990.
- [16] L. Infeld and T. E. Hull, The factorization method. *Rev. Modern Physics* **23** (1951), 21–68.
- [17] R. Koekoek and R. F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its q -analogue. *Reports of the Faculty of Technical Mathematics and Informatics No. 98-17*. Delft University of Technology, Delft, 1998.
- [18] M. Lorente, Raising and lowering operators, factorization method and differential/difference operators of hypergeometric type. *Preprint* 2000.
- [19] A. J. Macfarlane, On q -analogues of the quantum harmonic oscillator and the quantum group $SU_q(2)$. *J. Phys. A.* **22** (1989), 4581-4588.
- [20] W. Miller (Jr.), Lie theory and difference equations. I. *J. Math. Anal. Appl.* **28** (1969), 383–399.
- [21] W. Miller (Jr.), Lie theory and q -difference equations. *SIAM J. Math. Anal.* **1** (1970), 171–188.
- [22] Sh. M. Nagiyev, Difference Schrödinger equation and q -oscillator model, *Theoret. and Math. Phys.* **102** (1995), 180-187.
- [23] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*. Springer Series in Computational Physics. Springer-Verlag, Berlin, 1991.
- [24] A. F. Nikiforov and V. B. Uvarov, Polynomial Solutions of hypergeometric type difference equations and their classification. *Integral Transforms and Special Functions.* **1** (1993), 223-249.
- [25] E. Schrödinger, A method of determining quantum-mechanical eigenvalues and eigenfunctions. *Proc. Roy. Irish. Acad.* **46A** (1940), 9-16.
- [26] E. Schrödinger, The factorization of the hypergeometric equation. *Proc. Roy. Irish. Acad.* **47A** (1941), 53-54.
- [27] Yu. F. Smirnov, On factorization and algebraization of difference equation of hypergeometric type. In *Proc. Internat. Workshop on Orthogonal Polynomials in Mathematical Physics*, M. Alfaro et al. Eds. Servicio de Publicaciones de la Universidad Carlos III de Madrid, (1997) 153-161.
- [28] Yu. F. Smirnov, Finite difference equations and factorization method. In *Proc. 5th Wigner Symposium*, P. Kasperkovitz and D. Grau Eds. World Scientific (1998) 148-150.
- [29] Yu. F. Smirnov, Factorization method: New aspects. *Rev. Mex. Fis.* **45** (1999), 1-6.

- [30] V. Spiridonov, L. Vinet, A. Zhedanov, Difference Schrödinger operators with linear and exponential discrete spectra. *Lett. Math. Phys.* **29** (1993), 63–73.
- [31] N. Ja. Vilenkin and A.U. Klimyk, *Representations of Lie Groups and Special Functions*. **Vol. I,II,III**. Kluwer Academic Publishers. Dordrecht, 1992.